# LECTURE NO 19

Electrostatics

# TOPIC COVERED

- Energy density in the electrostatic field

To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges $Q_1$, $Q_2$, and $Q_3$ in an initially empty space shown shaded in Figure 4.22. No work is required to transfer $Q_1$ from infinity to $P_1$ because the space is initially charge free and there is no electric field [from eq. (4.59), $W = 0$]. The work done in transferring $Q_2$ from infinity to $P_2$ is equal to the product of $Q_2$ and the potential $V_{21}$ at $P_2$ due to $Q_1$. Similarly, the work done in positioning $Q_3$ at $P_3$ is equal to $Q_3(V_{32} + V_{31})$, where $V_{32}$ and $V_{31}$ are the potentials at $P_3$ due to $Q_2$ and $Q_1$, respectively. Hence the total work done in positioning the three charges is

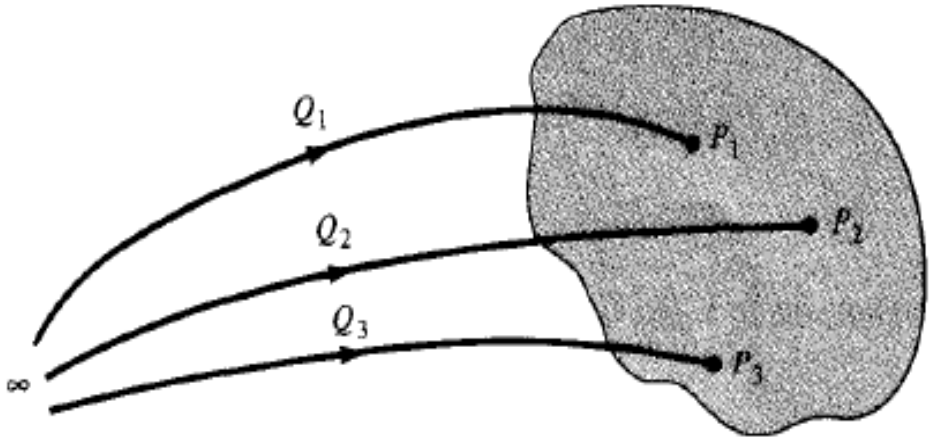
$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \tag{4.84}$$

If the charges were positioned in reverse order,

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13}) \tag{4.85}$$

where $V_{23}$ is the potential at $P_2$ due to $Q_3$, $V_{12}$ and $V_{13}$ are, respectively, the potentials at $P_1$ due to $Q_2$ and $Q_3$. Adding eqs. (4.84) and (4.85) gives

$$2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$$
$$= Q_1V_1 + Q_2V_2 + Q_3V_3$$




or

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \tag{4.86}$$

where $V_1$, $V_2$, and $V_3$ are total potentials at $P_1$, $P_2$, and $P_3$, respectively. In general, if there are $n$ point charges, eq. (4.86) becomes

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k} \quad \text{(in joules)} \tag{4.87}$$

If, instead of point charges, the region has a continuous charge distribution, the summation in eq. (4.87) becomes integration; that is,

$$W_E = \frac{1}{2}\int \rho_L V \, dl \qquad \text{(line charge)} \tag{4.88}$$

$$W_E = \frac{1}{2}\int \rho_S V \, dS \qquad \text{(surface charge)} \tag{4.89}$$

$$W_E = \frac{1}{2}\int \rho_v V \, dv \qquad \text{(volume charge)} \tag{4.90}$$

Since $\rho_v = \nabla \cdot \mathbf{D}$, eq. (4.90) can be further developed to yield

$$W_E = \frac{1}{2} \int_v (\nabla \cdot \mathbf{D})\, V\, dv \tag{4.91}$$

But for any vector $\mathbf{A}$ and scalar $V$, the identity

$$\nabla \cdot V\mathbf{A} = \mathbf{A} \cdot \nabla V + V(\nabla \cdot \mathbf{A})$$

or

$$(\nabla \cdot \mathbf{A})V = \nabla \cdot V\mathbf{A} - \mathbf{A} \cdot \nabla V \tag{4.92}$$

holds. Applying the identity in eqs. (4.92) to (4.91), we get

$$W_E = \frac{1}{2} \int_v (\nabla \cdot V\mathbf{D})\, dv - \frac{1}{2} \int_v (\mathbf{D} \cdot \nabla V)\, dv \tag{4.93}$$

By applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2}\int_v (\mathbf{D} \cdot \nabla V)\, dv \tag{4.94}$$

From Section 4.9, we recall that $V$ varies as $1/r$ and $\mathbf{D}$ as $1/r^2$ for point charges; $V$ varies as $1/r^2$ and $\mathbf{D}$ as $1/r^3$ for dipoles; and so on. Hence, $V\mathbf{D}$ in the first term on the right-hand side of eq. (4.94) must vary at least as $1/r^3$ while $dS$ varies as $r^2$. Consequently, the first integral in eq. (4.94) must tend to zero as the surface $S$ becomes large. Hence, eq. (4.94) reduces to

$$W_E = -\frac{1}{2}\int_v (\mathbf{D} \cdot \nabla V)\, dv = \frac{1}{2}\int_v (\mathbf{D} \cdot \mathbf{E})\, dv \tag{4.95}$$

and since $\mathbf{E} = -\nabla V$ and $\mathbf{D} = \varepsilon_o \mathbf{E}$

$$\boxed{W_E = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\int \varepsilon_o E^2\, dv} \tag{4.96}$$

# Assignment

Derive the expression for the energy density in the electrostatic field is given by